

# TRIPLE MASSEY PRODUCTS AND ABSOLUTE GALOIS GROUPS

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**ABSTRACT.** Let  $p$  be a prime number,  $F$  a field containing a root of unity of order  $p$ , and  $G_F$  the absolute Galois group. Extending results of Hopkins, Wickelgren, Mináč and Tân, we prove that the triple Massey product  $H^1(G_F)^3 \rightarrow H^2(G_F)$  contains 0 whenever it is nonempty. This gives a new restriction on the possible profinite group structure of  $G_F$ .

A main problem in modern Galois theory is to understand the group-theoretic structure of absolute Galois groups  $G_F = \text{Gal}(F_{\text{sep}}/F)$  of fields  $F$ , that is, the possible symmetry patterns of roots of polynomials. General restrictions on the possible structure of the profinite group  $G_F$  are rare: By classical results of Artin and Schreier, the torsion in  $G_F$  can consist only of involutions. In addition, the celebrated work of Voevodsky and Rost ([Voe03], [Voe11]) identifies the cohomology ring  $H^*(G_F) = H^*(G_F, \mathbb{Z}/m)$  with the mod- $m$  Milnor  $K$ -ring  $K_*^M(F)/m$ , assuming existence of  $m$ -th roots of unity. In particular, the graded ring  $H^*(G_F)$  is generated by its degree 1 elements, and its relations originate from the degree 2 component. This can be used to rule out many more profinite groups from being absolute Galois groups of fields ([CEM12], [EM11b]). In fact, the Artin–Schreier restriction about the torsion also follows from the latter results [EM11b, Ex. 6.4(2)].

Very recently, a remarkable series of works by Hopkins, Wickelgren, Mináč and Tân indicated the possible existence of a new kind of general restrictions on the structure of absolute Galois groups, related to the differential graded algebra  $C^*(G_F) = C^*(G_F, \mathbb{Z}/m)$  of continuous cochains on  $G_F$ . The interplay between  $C^*(G_F)$  and its cohomology algebra  $H^*(G_F)$  gives rise to *external* operations on  $H^*(G_F)$ , in addition to its (“internal”) ring structure with respect to the cup product, notably, the  *$n$ -fold Massey*

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products  $H^1(G_F)^n \rightarrow H^2(G_F)$ . The definition of the Massey product in the context of general differential algebras is recalled in §1, and at this stage we only mention that it is a multi-valued map, which for  $n = 2$  coincides with the cup product. The Massey product  $\langle \chi_1, \dots, \chi_n \rangle \subseteq H^2(G_F)$  is *essential* if it is non-empty, but does not contain 0. The above-mentioned works show that, under various assumptions, the *triple* Massey product for  $H^*(G_F)$  is never essential. Thus profinite groups  $G$  for which  $H^*(G)$  contains an essential triple Massey product cannot be realized as absolute Galois groups of fields satisfying these assumptions. In [MT14a] Mináč and Tân develop a method to produce such groups  $G$ , by examining their presentation by generators and relations modulo the 4th term in the  $p$ -Zassenhaus filtration. As a concrete example, the profinite group  $G$  on 5 generators  $\sigma_1, \dots, \sigma_5$  and the single defining relation  $[\sigma_4, \sigma_5][[\sigma_2, \sigma_3], \sigma_1]$  gives rise to an essential triple Massey product [MT14a, Ex. 7.2].

Specifically, assume that  $m = p$  is prime, and  $F$  contains a root of unity of order  $p$  (so  $\text{char } F \neq p$ ). It was shown that the triple Massey product for  $H^*(G_F)$  is never essential in the following situations:

- 1)  $p = 2$  and  $F$  is a local field or a global field (Hopkins and Wickelgren [HW15]);
- 2)  $p = 2$  and  $F$  is arbitrary (Mináč and Tân [MT14a]);
- 3)  $p$  is arbitrary and  $F$  is a local field (Mináč and Tân; follows from [MT14a, Th. 4.3] and [MT15, Th. 8.5]);
- 4)  $p$  is arbitrary, and  $F$  is a global field (Mináč and Tân [MT14b]).

Moreover, it is conjectured in [MT15] that the  $n$ -fold Massey product above is never essential for every  $n \geq 3$ . Also, in [EM15] we find close connections between these results and classical facts in the theory of central simple algebras. In particular, 2) is closely related to Albert's characterization from 1939 [Alb39] (as refined by Tignol [Tig79]; see also Rowen [Row84] and [Tig81]) of the central simple algebras of exponent 2 and degree 4 as biquaternionic algebras.

Motivated by these works, we prove in this paper the above conjecture for triple Massey products for arbitrary  $p$  and general fields  $F$  as above:

**Main Theorem.** *Let  $F$  be a field containing a root of unity of order  $p$ , and let  $\chi_1, \chi_2, \chi_3 \in H^1(G_F)$ . Then  $\langle \chi_1, \chi_2, \chi_3 \rangle$  is not essential.*

The Main Theorem was first proved by the second-named author using methods from the theory of central simple algebras, notably the Amitsur–Saltman theory of abelian crossed products [Mat14]. The current paper, which replaces [Mat14], is based on a shortcut which allows carrying

the original crossed product computations to the framework of profinite group cohomology (see Proposition 5.3). We also work in a more general formal context, and prove the Main Theorem for *p-Kummer formations*  $(G, A, \{\kappa_U\}_U)$  (Theorem 5.4). These structures axiomatize the relevant Galois-theoretic properties of absolute Galois groups: the Kummer isomorphism, Hilbert's Theorem 90, and the connections between restriction, corestriction, and cup product. The Main Theorem is just the case where  $G = G_F$ ,  $A = F_{\text{sep}}^\times$ , and the  $\kappa_U$  are the Kummer maps (see §5).

The Main Theorem is in a partial analogy with the important work of Deligne, Griffiths, Morgan, and Sullivan [DGMS75], which proves that any compact Kähler manifold is formal. This implies that its  $n$ -fold Massey products, with  $n \geq 3$ , are non-essential in the de Rahm context (see also [Huy05, Ch. 3.A]). On the other hand, links in  $\mathbb{R}^3$  provide examples of essential Massey products in the algebra of singular cochains. For instance, the *Borromean rings* give rise to an essential triple Massey product [Hil12, §10.1], and this explains why they are not equivalent to three unconnected circles. Thus the Main Theorem means that a phenomena such as the Borromean rings is impossible in this Galois cohomology context. We also note that examples due to Positselski show that  $H^*(G_F)$  may not be formal ([Pos11, §9.11], [Pos15]).

Among the other works on Massey products in Galois cohomology we mention those by Morishita [Mor04], Sharifi ([Sha99], [Sha07]), Wickelgren ([Wic12a], [Wic12b]), Vogel [Vog05], Gärtner [Gär15], and the first-named author [Efr14].

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*Addendum (January 2015):* In the recent paper [MT14c] (which was posted after the initial version [Mat14] of the current work) Mináč and Tân also give a Galois-cohomological proof of the Main Theorem, which is similar at several points to our proof; See also [MT15]. Moreover, they point out that the standard restriction-corestriction argument allows one to remove the assumption that the field contains a root of unity of order  $p$ . Namely, for a  $p$ th root of unity  $\zeta$ , the index of  $U = G_{F(\zeta)}$  in  $G = G_F$  is prime to  $p$ . If  $\chi_1, \chi_2, \chi_3 \in H^1(G)$  and  $\alpha \in \langle \chi_1, \chi_2, \chi_3 \rangle$ , then by our Main Theorem,  $\text{Res}_U \alpha = \text{Res}_U(\chi_1) \cup \psi_1 + \text{Res}_U(\chi_3) \cup \psi_3$  for some  $\psi_1, \psi_3 \in H^1(U)$ . Hence

$(G: U)\alpha = \chi_1 \cup \text{Cor}_G(\psi_1) + \chi_3 \cup \text{Cor}_G(\psi_3)$ , and consequently  $0 \in \langle \chi_1, \chi_2, \chi_3 \rangle$  (see §1).

## 1. MASSEY PRODUCTS

We recall the definition and basic properties of Massey products of degree 1 cohomology elements. We first recall that a **differential graded algebra** over a ring  $R$  (abbreviated  $R$ -DGA) is a graded  $R$ -algebra  $C^\bullet = \bigoplus_{r=0}^{\infty} C^r$  equipped with  $R$ -module homomorphisms  $\partial^s: C^r \rightarrow C^{r+1}$  such that  $\partial = \bigoplus_{r=0}^{\infty} \partial^r$  satisfies  $\partial \circ \partial = 0$  and one has:  $\partial^{r+s}(ab) = \partial^r(a)b + (-1)^r a \partial^s(b)$  for  $a \in C^r$ ,  $b \in C^s$  (the *Leibnitz rule*). Set  $Z^r = \text{Ker}(\partial^r)$ ,  $B^r = \text{Im}(\partial^{r-1})$ , and  $H^r = Z^r/B^r$ , and let  $[c]$  denote the class of  $c \in Z^r$  in  $H^r$ . Then  $H^\bullet = \bigoplus_{r=0}^{\infty} H^r$  has an induced  $R$ -DGA structure with zero differentials  $\partial^r$ . We say that the DGA  $C^\bullet$  is **graded-commutative** if  $ab = (-1)^{rs}ba$  for  $a \in C^r$  and  $b \in C^s$ .

We fix an integer  $n \geq 2$ . Consider a system  $c_{ij} \in C^1$ , where  $1 \leq i \leq j \leq n$  and  $(i, j) \neq (1, n)$ . For any  $i, j$  satisfying  $1 \leq i \leq j \leq n$  (including  $(i, j) = (1, n)$ ) we define

$$\widetilde{c}_{ij} = \sum_{r=i}^{j-1} c_{ir} c_{r+1,j} \in C^2.$$

One says that  $(c_{ij})$  is a **defining system of size  $n$**  in  $C^\bullet$  if  $\partial c_{ij} = \widetilde{c}_{ij}$  for every  $1 \leq i \leq j \leq n$  with  $(i, j) \neq (1, n)$ . We also say that the defining system  $(c_{ij})$  is **on**  $c_{11}, \dots, c_{nn}$ . Note that then  $c_{ii}$  is a 1-cocycle,  $i = 1, 2, \dots, n$ . Further,  $\widetilde{c}_{1n}$  is a 2-cocycle ([Kra66, p. 432], [Fen83, p. 233]). Its cohomology class depends only on the cohomology classes  $[c_{11}], \dots, [c_{nn}]$  [Kra66, Th. 3]. Given  $c_1, \dots, c_n \in Z^1$ , the  **$n$ -fold Massey product** of  $\langle [c_1], \dots, [c_n] \rangle$  is the subset of  $H^2$  consisting of all cohomology classes  $[\widetilde{c}_{1n}]$  obtained from defining systems  $(c_{ij})$  of size  $n$  on  $c_1, \dots, c_n$  in  $C^\bullet$ . The Massey product  $\langle [c_1], \dots, [c_n] \rangle$  is **essential** if it is non-empty but does not contain 0.

When  $n = 2$ ,  $\langle [c_1], [c_2] \rangle$  is always non-empty and consists only of  $[c_1][c_2]$ . In the case  $n = 3$  one has the following well-known facts:

**Proposition 1.1** ([EM15, Prop. 6.1]). *Let  $c_1, c_2, c_3 \in Z^1$ .*

- (a)  $\langle [c_1], [c_2], [c_3] \rangle$  is non-empty if and only if  $[c_1][c_2] = [c_2][c_3] = 0$ ;
- (b) If  $(c_{ij})$  is a defining system on  $[c_1], [c_2], [c_3]$ , then  $\langle [c_1], [c_2], [c_3] \rangle = [\widetilde{c}_{13}] + [c_1]H^1 + H^1[c_3]$ .

## 2. COHOMOLOGICAL PRELIMINARIES

We refer, e.g., to [NSW08] for the basic notions and facts in profinite and Galois cohomology. Let  $p$  be a fixed prime number and let  $G$  be a profinite group acting trivially on  $\mathbb{Z}/p$ . We write  $C^r(G)$  for the group  $C^r(G, \mathbb{Z}/p)$  of continuous (inhomogenous) cochains  $G^r \rightarrow \mathbb{Z}/p$ . Let  $Z^r(G) = Z^r(G, \mathbb{Z}/p)$  and  $B^r(G) = B^r(G, \mathbb{Z}/p)$  be its subgroups of  $r$ -cocycles and  $r$ -coboundaries, respectively, and let  $H^r(G) = H^r(G, \mathbb{Z}/p)$  be the corresponding profinite cohomology group. We identify  $H^1(G) = \text{Hom}(G, \mathbb{Z}/p)$ . Then  $C^\bullet(G) = \bigoplus_{r=0}^\infty C^r(G)$  is a DGA over  $\mathbb{F}_p$  with the cup product  $\cup$ . Its cohomology DGA  $H^\bullet(G) = \bigoplus_{r=0}^\infty H^r(G)$  is graded-commutative. We will need the following slightly refined version of this property for degree 1 elements:

**Lemma 2.1.** Let  $\chi_1, \chi_2 \in H^1(G)$ . Then there exists  $\psi \in C^1(G)$  such that  $\partial\psi = \chi_1 \cup \chi_2 + \chi_2 \cup \chi_1$  and  $\psi$  is zero on  $\text{Ker}(\chi_i)$ ,  $i = 1, 2$ .

*Proof.* When  $\chi_1, \chi_2$  are  $\mathbb{F}_p$ -linearly independent, let  $\bar{G} = G/(\text{Ker}(\chi_1) \cap \text{Ker}(\chi_2)) \cong (\mathbb{Z}/p)^2$ , and choose  $\bar{\sigma}_1, \bar{\sigma}_2 \in \bar{G}$  which are dual to  $\chi_1, \chi_2$ . Define  $\bar{\psi} \in C^1(\bar{G})$  by  $\bar{\psi}(\bar{\sigma}_1^i \bar{\sigma}_2^j) = -ij$  for  $0 \leq i, j < p$ , and take  $\psi = \text{Inf}_G \bar{\psi}$  be its inflation to  $H^1(G)$ .

When  $\chi_1, \chi_2$  are nonzero and  $\mathbb{F}_p$ -linearly dependent, we write  $\chi_2 = k\chi_1$  with  $1 \leq k < p$  and  $\bar{G} = G/\text{Ker}(\chi_1) \cong \mathbb{Z}/p$ . We define  $\bar{\psi} \in C^1(\bar{G})$  by  $\bar{\psi}(\bar{\sigma}_1^i) = -ki^2 \in \mathbb{Z}/p$ , and take  $\psi = \text{inf}_G \bar{\psi}$ .

Finally, when at least one of  $\chi_1, \chi_2$  is 0 we take  $\psi = 0 \in C^1(G)$ .  $\square$

Given a closed subgroup  $U$  of  $G$  let  $\text{Res}_U: H^i(G) \rightarrow H^i(U)$  be the restriction homomorphism. When  $U$  is open in  $G$  we have a corestriction homomorphism  $\text{Cor}_G: H^i(U) \rightarrow H^i(G)$ . If  $N$  is a closed normal subgroup of  $G$ , then every  $\sigma \in G$  induces a homomorphism  $\sigma: H^1(N) \rightarrow H^1(N)$ ,  $\varphi \mapsto \sigma\varphi$ , where  $(\sigma\varphi)(\tau) = \varphi(\sigma\tau\sigma^{-1})$ .

For a closed subgroup  $U$  of  $G$  and for  $\chi \in H^1(U)$ , we consider the sequence:

$$(2.1) \quad H^1(\text{Ker}(\chi)) \xrightarrow{\text{Cor}_U} H^1(U) \xrightarrow{\chi \cup} H^2(U) \xrightarrow{\text{Res}_{\text{Ker}(\chi)}} H^2(\text{Ker}(\chi)).$$

**Example 2.2.** When  $G = G_F$  for a field  $F$  containing a root of unity of order  $p$ , this sequence is exact for every such  $U$  and  $\chi$ . This corresponds to the isomorphism  $K^\times/N_{L/K}(L^\times) \cong \text{Br}(L/K)$  for the fixed fields  $K, L$  of  $U, \text{Ker}(\chi)$ , respectively, where  $\text{Br}(L/K)$  is the relative Brauer group of the field extension  $L \supseteq K$  [Dra83, p. 73, Th. 1].

**Proposition 2.3.** *Suppose that (2.1) with  $U = G$  is exact at  $H^2(G)$  for every  $\chi \in H^1(G)$ . For every  $\chi_1, \chi_2, \chi_3 \in H^1(G)$  one has  $\langle \chi_1, \chi_2, \chi_3 \rangle = \langle \chi_3, \chi_2, \chi_1 \rangle$ .*

*Proof.* Since both Massey products are cosets of  $\chi_1 \cup H^1(G) + \chi_3 \cup H^1(G)$  (Proposition 1.1(b)), it suffices to show that  $\langle \chi_1, \chi_2, \chi_3 \rangle \supseteq \langle \chi_3, \chi_2, \chi_1 \rangle$ . So let  $\alpha \in \langle \chi_3, \chi_2, \chi_1 \rangle$ . Then there exist  $\varphi_{32}, \varphi_{21} \in C^1(G)$  such that

$$\partial\varphi_{32} = \chi_3 \cup \chi_2, \quad \partial\varphi_{21} = \chi_2 \cup \chi_1, \quad \alpha = [\chi_3 \cup \varphi_{21} + \varphi_{32} \cup \chi_1].$$

Let  $K = \text{Ker}(\chi_1)$ . Lemma 2.1 yields  $\psi_{12} \in C^1(G)$  such that  $\partial\psi_{12} = \chi_1 \cup \chi_2 + \chi_2 \cup \chi_1$  in  $C^2(G)$  and  $\psi_{12} = 0$  on  $K = \text{Ker}(\chi_1)$ . The graded-commutativity of  $H^\bullet(G)$  yields  $\psi_{23} \in C^1(G)$  such that  $\partial\psi_{23} = \chi_2 \cup \chi_3 + \chi_3 \cup \chi_2$  in  $C^2(G)$ . Taking  $\varphi_{12} = \psi_{12} - \varphi_{21}$  and  $\varphi_{23} = \psi_{23} - \varphi_{32}$ , we obtain that  $\partial\varphi_{12} = \chi_1 \cup \chi_2$  and  $\partial\varphi_{23} = \chi_2 \cup \chi_3$ . It therefore suffices to show that  $[\chi_1 \cup \varphi_{23} + \varphi_{12} \cup \chi_3]$  and  $\alpha$  are equal modulo the indeterminacy  $\chi_1 \cup H^1(G) + \chi_3 \cup H^1(G)$  of both Massey products.

Now  $\text{Res}_K(\partial\varphi_{21}) = \text{Res}_K(\chi_2 \cup \chi_1) = 0$ , so  $\text{Res}_K \varphi_{21} \in Z^1(K)$ . The graded-commutativity of  $H^\bullet(K)$  gives  $\text{Res}_K(\varphi_{21} \cup \chi_3 + \chi_3 \cup \varphi_{21}) \in B^2(K)$ . As  $\text{Res}_K \psi_{12} = 0$  we obtain that

$$\begin{aligned} \text{Res}_K(\chi_1 \cup \varphi_{23} + \varphi_{12} \cup \chi_3) &= \text{Res}_K(\varphi_{12} \cup \chi_3) = -\text{Res}_K(\varphi_{21} \cup \chi_3) \\ &\equiv \text{Res}_K(\chi_3 \cup \varphi_{21}) = \text{Res}_K(\chi_3 \cup \varphi_{21} + \varphi_{32} \cup \chi_1) \pmod{B^2(K)}. \end{aligned}$$

Hence  $\text{Res}_K[\chi_1 \cup \varphi_{23} + \varphi_{12} \cup \chi_3] = \text{Res}_K \alpha$ . By (2.1),

$$\alpha - [\chi_1 \cup \varphi_{23} + \varphi_{12} \cup \chi_3] \in \chi_1 \cup H^1(G),$$

as desired.  $\square$

**Remark 2.4.** Vogel [Vog04, Example 1.2.11] proves the assertion of Proposition 2.3 under the assumption that  $G = F/R$  for a free pro- $p$  group  $F$  and a closed normal subgroup  $R$  of  $F$  contained in the third term of its lower central sequence. In a topological context, Kraines [Kra66, Th. 8] proves that Massey products of arbitrary length remain the same up to a sign when the order of the entries is reversed.

**Proposition 2.5.** *Suppose that (2.1) with  $U = G$  is exact at  $H^2(G)$  for every  $\chi \in H^1(G)$ . The following conditions are equivalent:*

- (1) *For every  $\chi_1, \chi_2, \chi_3 \in H^1(G)$ , the Massey product  $\langle \chi_1, \chi_2, \chi_3 \rangle$  is not essential.*
- (2) *For every  $\chi_1, \chi_2, \chi_3 \in H^1(G)$  such that the pairs  $\chi_1, \chi_3$  and  $\chi_2, \chi_3$  are  $\mathbb{F}_p$ -linearly independent,  $\langle \chi_1, \chi_2, \chi_3 \rangle$  is not essential.*

*Proof.* (1) $\Rightarrow$ (2): Trivial.

(2) $\Rightarrow$ (1): Suppose that  $\langle \chi_1, \chi_2, \chi_3 \rangle \neq \emptyset$ . By Proposition 1.1(a),  $\chi_1 \cup \chi_2 = 0 = \chi_2 \cup \chi_3$  in  $H^2(G)$ . Therefore there exist  $\varphi_{12}, \varphi_{23} \in C^1(G)$  such that  $\partial\varphi_{12} = \chi_1 \cup \chi_2$  and  $\partial\varphi_{23} = \chi_2 \cup \chi_3$  in  $C^2(G)$ . Then  $\chi_1 \cup \varphi_{23} + \varphi_{12} \cup \chi_3 \in Z^2(G)$ . By Proposition 1.1(b), we need to find  $\varphi_{12}, \varphi_{23}$  such that the cohomology class of this 2-cocycle is contained in the subset  $\chi_1 \cup H^1(G) + \chi_3 \cup H^1(G)$  of  $H^2(G)$ . We break the discussion into several cases.

Case I: *The pairs  $\chi_1, \chi_3$  and  $\chi_2, \chi_3$  are  $\mathbb{F}_p$ -linearly independent.* Then we simply apply (2).

Case II:  *$\chi_1, \chi_3$  are  $\mathbb{F}_p$ -linearly dependent.* We may assume that  $\chi_1 = i\chi_3$  for some  $i \in \mathbb{F}_p$ . Given  $\varphi_{12}, \varphi_{23}$  as above we then have

$$\text{Res}_{\text{Ker}(\chi_3)}(\chi_1 \cup \varphi_{23} + \varphi_{12} \cup \chi_3) = 0.$$

By (2.1),  $[\chi_1 \cup \varphi_{23} + \varphi_{12} \cup \chi_3] \in \chi_3 \cup H^1(G)$ , and we are done.

Case III:  $\chi_2 = 0$ . Then  $\chi_1 \cup \chi_2 = 0 = \chi_2 \cup \chi_3$  in  $C^2(G)$ , so for  $\varphi_{12} = \varphi_{23} = 0$  we have  $[\chi_1 \cup \varphi_{23} + \varphi_{12} \cup \chi_3] = 0$ .

Case IV:  *$\chi_1, \chi_3$  are  $\mathbb{F}_p$ -linearly independent,  $\chi_2 \neq 0$ , and  $\chi_2, \chi_3$  are  $\mathbb{F}_p$ -linearly dependent.* Then  $\chi_1, \chi_2$  are also  $\mathbb{F}_p$ -independent. By Proposition 2.3,  $\langle \chi_1, \chi_2, \chi_3 \rangle = \langle \chi_3, \chi_2, \chi_1 \rangle$ , and by (2),  $\langle \chi_3, \chi_2, \chi_1 \rangle$  is not essential.  $\square$

### 3. CUP PRODUCTS AS COBOUNDARIES

Let  $G$  be a profinite group and let  $\chi_a, \chi_b \in H^1(G)$  be  $\mathbb{F}_p$ -linearly independent. Set  $N_a = \text{Ker}(\chi_a)$ ,  $N_b = \text{Ker}(\chi_b)$  and  $L = N_a \cap N_b$ . Thus  $G/L \cong (G/N_a) \times (G/N_b) \cong (\mathbb{Z}/p)^2$ . Let  $\sigma_a, \sigma_b \in G$  be dual to  $\chi_1, \chi_b$ , respectively, i.e.,

$$\chi_a(\sigma_a) = 1, \quad \chi_a(\sigma_b) = 0, \quad \chi_b(\sigma_a) = 0, \quad \chi_b(\sigma_b) = 1.$$

Let  $\tau = [\sigma_a, \sigma_b] = \sigma_a \sigma_b \sigma_a^{-1} \sigma_b^{-1}$ .

**Proposition 3.1.** *Suppose that  $\omega \in H^1(N_b)$  satisfies  $\omega - \sigma_b \omega = \text{Res}_{N_b} \chi_a$ . Then*

- (a)  $\omega(\tau) = 1$ ;
- (b)  $N_a \cap \text{Ker}(\omega)$  is normal in  $G$ ;
- (c)  $(G : N_a \cap \text{Ker}(\omega)) = p^3$ ;
- (d) *The images  $\bar{\sigma}_a, \bar{\sigma}_b, \bar{\tau}$  of  $\sigma_a, \sigma_b, \tau$ , respectively, in  $\bar{G} = G/(N_a \cap \text{Ker}(\omega))$  generate  $\bar{G}$  and satisfy  $[\bar{\tau}, \bar{\sigma}_a] = [\bar{\tau}, \bar{\sigma}_b] = 1$ .*

*Proof.* (a) Since  $\sigma_a, \sigma_b \sigma_a \sigma_b^{-1} \in N_b$ , the assumption on  $\omega$  gives

$$\omega(\tau) = \omega(\sigma_a) + \omega(\sigma_b \sigma_a^{-1} \sigma_b^{-1}) = \omega(\sigma_a) - (\sigma_b \omega)(\sigma_a) = (\text{Res}_{N_b} \chi_a)(\sigma_a) = 1.$$

(b) For every  $\sigma \in N_b$  we have  $\sigma\omega = \omega$ , and therefore  $\sigma(\text{Res}_L \omega) = \text{Res}_L \omega$ . By the assumption on  $\omega$ ,  $\text{Res}_L \omega - \sigma_b(\text{Res}_L \omega) = \text{Res}_L \chi_a = 0$ . Therefore  $\sigma(\text{Res}_L \omega) = \text{Res}_L \omega$  for every  $\sigma \in \langle N_b, \sigma_b \rangle = G$ . This means that  $\omega(\sigma h \sigma^{-1}) = \omega(h)$  for every  $\sigma \in G$  and  $h \in L$ . Consequently,  $\text{Ker}(\text{Res}_L \omega)$  is normal in  $G$ , and we observe that  $N_a \cap \text{Ker}(\omega) = \text{Ker}(\text{Res}_L \omega)$ .

(c) We note that every commutator in  $G$  is contained in  $L$ . By this and (a),  $\tau \in L \setminus \text{Ker}(\text{Res}_L \omega)$ , whence  $(L : \text{Ker}(\text{Res}_L \omega)) = p$ . Consequently,

$$(G : N_a \cap \text{Ker}(\omega)) = (G : L)(L : \text{Ker}(\text{Res}_L \omega)) = p^2 \cdot p = p^3.$$

(d) The images of  $\bar{\sigma}_a, \bar{\sigma}_b$  generate  $G/L \cong (\mathbb{Z}/p)^2$ . Also,  $L/(N_a \cap \text{Ker}(\omega)) = L/\text{Ker}(\text{Res}_L \omega)$  is generated by  $\bar{\tau}$ , by (a). Hence  $\bar{\sigma}_a, \bar{\sigma}_b, \bar{\tau}$  generate  $\bar{G}$ . Since  $\sigma_a, \tau \in N_b$  we have  $\omega(\tau \sigma_a \tau^{-1} \sigma_a^{-1}) = 0$ , so  $\tau \sigma_a \tau^{-1} \sigma_a^{-1} \in N_a \cap \text{Ker}(\omega)$ . Therefore  $[\bar{\tau}, \bar{\sigma}_a] = 1$ .

As  $\tau \in N_a \cap N_b$ ,

$$\omega(\tau \sigma_b \tau^{-1} \sigma_b^{-1}) = \omega(\tau) + (\sigma_b \omega)(\tau^{-1}) = \omega(\tau) - (\sigma_b \omega)(\tau) = (\text{Res}_{N_b} \chi_a)(\tau) = 0.$$

Therefore  $\tau \sigma_b \tau^{-1} \sigma_b^{-1} \in N_a \cap \text{Ker}(\omega)$ , i.e.,  $[\bar{\tau}, \bar{\sigma}_b] = 1$ .  $\square$

It follows from Proposition 3.1 that  $\bar{G}$  is the Heisenberg group  $H_{p^3}$  ( $D_4$  when  $p = 2$ ). We refer to [Sha99, Ch. II] for related results.

**Proposition 3.2.** *Suppose that  $\omega \in H^1(N_b)$  satisfies  $\omega - \sigma_b \omega = \text{Res}_{N_b} \chi_a$ . There exists  $\varphi \in C^1(G)$  with  $\partial\varphi = -\chi_a \cup \chi_b$  in  $C^2(G)$  and  $\omega = \text{Res}_{N_b} \varphi$  in  $C^1(N_b)$ .*

*Proof.* Let  $\bar{\chi}_a, \bar{\chi}_b \in Z^1(\bar{G})$  be the characters with inflations  $\chi_a, \chi_b$ , respectively, to  $G$ . Every element of  $\bar{G}$  can be uniquely written as  $\bar{\sigma}_b^i \bar{\sigma}_a^j \bar{\tau}^k$  for integers  $0 \leq i, j, k < p$  (which we also consider as elements of  $\mathbb{Z}/p$ ). We define  $\bar{\varphi} \in C^1(\bar{G})$  by  $\bar{\varphi}(\bar{\sigma}) = \omega(\sigma_a)j + k$ . Let  $\varphi \in C^1(G)$  be the inflation of  $\bar{\varphi}$  to  $G$ .

To compute  $\partial\varphi$ , we take  $0 \leq i, j, k, r, s, t < p$ . Then  $\bar{\sigma}_a^j \bar{\sigma}_b^r = \bar{\sigma}_b^r \bar{\sigma}_a^j \bar{\tau}^{jr}$ , so

$$\bar{\varphi}(\bar{\sigma}_b^i \bar{\sigma}_a^j \bar{\tau}^k \bar{\sigma}_b^r \bar{\sigma}_a^s \bar{\tau}^t) = \bar{\varphi}(\bar{\sigma}_b^{i+r} \bar{\sigma}_a^{j+s} \bar{\tau}^{k+t+jr}) = \omega(\sigma_a)(j+s) + k + t + jr.$$

Therefore

$$\begin{aligned} (\partial\bar{\varphi})(\bar{\sigma}_b^i \bar{\sigma}_a^j \bar{\tau}^k, \bar{\sigma}_b^r \bar{\sigma}_a^s \bar{\tau}^t) &= \bar{\varphi}(\bar{\sigma}_b^i \bar{\sigma}_a^j \bar{\tau}^k) + \bar{\varphi}(\bar{\sigma}_b^r \bar{\sigma}_a^s \bar{\tau}^t) - \bar{\varphi}(\bar{\sigma}_b^i \bar{\sigma}_a^j \bar{\tau}^k \bar{\sigma}_b^r \bar{\sigma}_a^s \bar{\tau}^t) \\ &= \omega(\sigma_a)j + k + \omega(\sigma_a)s + t - (\omega(\sigma_a)(j+s) + k + t + jr) = -jr \\ &= -\bar{\chi}_a(\bar{\sigma}_b^i \bar{\sigma}_a^j \bar{\tau}^k) \bar{\chi}_b(\bar{\sigma}_b^r \bar{\sigma}_a^s \bar{\tau}^t) = -(\bar{\chi}_a \cup \bar{\chi}_b)(\bar{\sigma}_b^i \bar{\sigma}_a^j \bar{\tau}^k, \bar{\sigma}_b^r \bar{\sigma}_a^s \bar{\tau}^t). \end{aligned}$$



The first equality of the Proposition now follows by inflation to  $G$ .

For the second equality, let  $\sigma \in N_b$  and let  $\bar{\sigma}$  be the image of  $\sigma$  in  $N_b/(N_a \cap \text{Ker}(\omega))$ . We may write  $\bar{\sigma} = \bar{\sigma}_a^j \bar{\tau}^k$  for some integers  $0 \leq j, k < p$ . Since  $\omega(\tau) = 1$  (Proposition 3.1(a)) we have

$$\omega(\sigma) = \omega(\sigma_a^j \tau^k) = \omega(\sigma_a)j + k = \varphi(\sigma). \quad \square$$

#### 4. MASSEY PRODUCTS CONTAINING 0

Let  $\chi_1, \chi_2, \chi_3 \in H^1(G)$ , and set  $N_1 = \text{Ker}(\chi_1)$ ,  $N_3 = \text{Ker}(\chi_3)$  and  $M = N_1 \cap N_3$ . Suppose that  $\sigma_3 \in G$  satisfies  $\chi_1(\sigma_3) = 0$  and  $\chi_3(\sigma_3) = 1$ . Also let  $\omega \in H^1(N_3)$ . We assume that

$$(4.1) \quad \omega - \sigma_3 \omega = \text{Res}_{N_3} \chi_2, \quad \chi_1 \cup \chi_2 = 0,$$

and  $\chi_2, \chi_3$  are  $\mathbb{F}_p$ -linearly independent.

**Lemma 4.1.** *The triple Massey product  $\langle \chi_1, \chi_2, \chi_3 \rangle$  has a representative  $\alpha$  such that  $\text{Res}_{N_3} \alpha = -\text{Res}_{N_3}(\chi_1) \cup \omega$ .*

*Proof.* Since  $\chi_1 \cup \chi_2 = 0$  in  $H^2(G)$  there exists  $\varphi_{12} \in C^2(G)$  such that  $\partial \varphi_{12} = \chi_1 \cup \chi_2$  in  $C^2(G)$ . Proposition 3.2 and (4.1) give rise to  $\varphi_{23} \in C^1(G)$  with  $\partial \varphi_{23} = -\chi_2 \cup \chi_3$  and  $\omega = \text{Res}_{N_3} \varphi_{23}$ . Then  $\chi_1 \cup (-\varphi_{23}) + \varphi_{12} \cup \chi_3$  is a 2-cocycle with cohomology class  $\alpha$  in  $\langle \chi_1, \chi_2, \chi_3 \rangle$ . We have

$$\text{Res}_{N_3}(\chi_1 \cup (-\varphi_{23}) + \varphi_{12} \cup \chi_3) = -\text{Res}_{N_3}(\chi_1) \cup \omega,$$

in  $C^2(N_3)$ , whence  $\text{Res}_{N_3} \alpha = -\text{Res}_{N_3}(\chi_1) \cup \omega$  in  $H^2(N_3)$ .  $\square$

**Theorem 4.2.** *In the above setup (4.1), assume further that the sequence (2.1) is exact for every open subgroup  $U$  of  $G$  of index dividing  $p$  and every  $\chi \in H^1(U)$ . Then the following conditions are equivalent:*

- (1)  $0 \in \langle \chi_1, \chi_2, \chi_3 \rangle$ ;
- (2) *There exists  $\lambda \in H^1(G)$  such that  $\text{Res}_{N_3}(\chi_1 \cup \lambda) = \text{Res}_{N_3}(\chi_1) \cup \omega$ ;*
- (3)  $\omega \in \text{Res}_{N_3} H^1(G) + \text{Cor}_{N_3} H^1(M)$ .

*Proof.* (1) $\Rightarrow$ (2): Lemma 4.1 yields  $\alpha \in \langle \chi_1, \chi_2, \chi_3 \rangle$  with  $\text{Res}_{N_3} \alpha = -\text{Res}_{N_3}(\chi_1) \cup \omega$ . Since also  $0 \in \langle \chi_1, \chi_2, \chi_3 \rangle$ , Proposition 1.1(b) gives  $\lambda, \lambda' \in H^1(G)$  such that  $-\alpha = \chi_1 \cup \lambda + \chi_3 \cup \lambda'$ . Now this implies that  $\text{Res}_{N_3} \alpha = -\text{Res}_{N_3}(\chi_1 \cup \lambda)$ , whence (2).

(2) $\Rightarrow$ (1): For  $\alpha$  as in Lemma 4.1,  $\text{Res}_{N_3}(\alpha + \chi_1 \cup \lambda) = 0$ . By the exact sequence (2.1),  $\alpha + \chi_1 \cup \lambda \in \chi_3 \cup H^1(G)$ , whence (1).

(2) $\Leftrightarrow$ (3): This follows again from (2.1).  $\square$

## 5. KUMMER FORMATIONS

Let  $A$  be a discrete  $G$ -module. For a closed normal subgroup  $U$  of  $G$  let  $A^U$  be the submodule of  $A$  fixed by  $U$ . There is an induced  $G/U$ -action on  $A^U$ .

For every open normal subgroups  $U \leq U'$  of  $G$  let  $N_{U'/U}: A^U \rightarrow A^{U'}$  be the trace map  $a \mapsto \sum_{\sigma} \sigma a$ , where  $\sigma$  ranges over a system of representatives for the cosets of  $U'$  modulo  $U$ .

Let  $I_{U'/U}$  be the subgroup of  $A^U$  consisting of all elements of the form  $\bar{\sigma}a - a$  with  $\bar{\sigma} \in U'/U$  and  $a \in A$ . We recall that

$$\hat{H}^{-1}(U'/U, A^U) = \text{Ker}(N_{U'/U})/I_{U'/U}.$$

When  $U'/U$  is cyclic with generator  $\bar{\sigma}$ , the subgroup  $I_{U'/U}$  consists of all elements  $\bar{\sigma}a - a$ , with  $a \in A$  (since  $\bar{\sigma}^k - 1 = (\bar{\sigma} - 1) \sum_{i=0}^{k-1} \bar{\sigma}^i$ ). Then  $\hat{H}^{-1}(U'/U, A^U) \cong H^1(U'/U, A^U)$  [NSW08, Prop. 1.7.1].

**Definition 5.1.** A  $p$ -**Kummer formation**  $(G, A, \{\kappa_U\}_U)$  consists of a profinite group  $G$ , a discrete  $G$ -module  $A$ , and for each open normal subgroup  $U$  of  $G$  a  $G$ -equivariant epimorphism  $\kappa_U: A^U \rightarrow H^1(U)$  such that for every open normal subgroup  $U$  of  $G$  the following conditions hold:

- (i) the sequence (2.1) is exact for every  $\chi \in H^1(U)$ ;
- (ii)  $\text{Ker}(\kappa_U) = pA^U$ ;
- (iii) for every open normal subgroup  $U'$  of  $G$  such that  $U \leq U'$ , there are commutative squares

$$\begin{array}{ccc} A^U & \xrightarrow{\kappa_U} & H^1(U) \\ \uparrow & & \uparrow \text{Res}_U \\ A^{U'} & \xrightarrow{\kappa_{U'}} & H^1(U') \end{array}; \quad \begin{array}{ccc} A^U & \xrightarrow{\kappa_U} & H^1(U) \\ N_{U'/U} \downarrow & & \downarrow \text{Cor}_{U'} \\ A^{U'} & \xrightarrow{\kappa_{U'}} & H^1(U') \end{array};$$

- (iv) for every open normal subgroup  $U'$  of  $G$  such that  $U \leq U'$  and  $(U' : U) = p$  one has  $\hat{H}^{-1}(U'/U, A^U) = 0$ .

**Example 5.2.** Let  $F$  be a field which contains a root of unity of order  $p$ . We fix an isomorphism between the group  $\mu_p$  of  $p$ th roots of unity and  $\mathbb{Z}/p$ . Given an open subgroup  $U$  of  $G_F$  let  $E = F_{\text{sep}}^U$  be its fixed field. The **Kummer homomorphism**  $\kappa_U: E^\times \rightarrow H^1(U)$  is the connecting homomorphism arising from the short exact sequence of  $U$ -modules

$$0 \rightarrow \mathbb{Z}/p \rightarrow F_{\text{sep}}^\times \xrightarrow{p} F_{\text{sep}}^\times \rightarrow 1.$$

By Hilbert's Theorem 90 it is surjective. Then  $(G_F, F_{\text{sep}}^\times, \{\kappa_U\}_U)$  is a  $p$ -Kummer formation. Indeed, (i) was pointed out in Example 2.2. (ii) is the

standard fact that  $\text{Ker}(\kappa_U) = (E^\times)^p$ , and (iii) follows from the commutativity of connecting homomorphisms with restrictions and corestrictions. For (iv) use the isomorphism  $\hat{H}^{-1}(U'/U, A^U) = H^1(U'/U, A^U)$  for  $U'/U$  cyclic and Hilbert's Theorem 90.

**Proposition 5.3.** *Let  $(G, A, \{\kappa_U\}_U)$  be a  $p$ -Kummer formation. Let  $M_1, M_3$  be distinct normal subgroups of  $G$  of index  $p$ , let  $M = M_1 \cap M_3$ , and let  $\sigma_3 \in M_1$  satisfy  $G = \langle M_3, \sigma_3 \rangle$ . Suppose that  $\lambda_1 \in H^1(M_1)$  and  $\lambda_3 \in H^1(M_3)$  satisfy  $\text{Cor}_G \lambda_1 = \text{Cor}_G \lambda_3$ . Then there exists  $\omega \in H^1(M_3)$  such that*

$$\sigma_3 \omega - \omega = -\text{Res}_{M_3} \text{Cor}_G \lambda_3, \quad \omega \in \text{Res}_{M_3} H^1(G) + \text{Cor}_{M_3} H^1(M).$$

*Proof.* There exist  $y_1 \in A^{M_1}$  and  $y_3 \in A^{M_3}$  such that  $\kappa_{M_1}(y_1) = \lambda_1$  and  $\kappa_{M_3}(y_3) = \lambda_3$ . Let  $w = \sum_{i=0}^{p-1} i \sigma_3^i y_3$ , and note that  $w \in A^{M_3}$ . We have  $(\sigma_3 - 1) \sum_{i=0}^{p-1} i \sigma_3^i = (p-1) \sigma_3^p + 1 - \sum_{i=0}^{p-1} \sigma_3^i$  in  $\mathbb{Z}[G]$ . As  $\sigma_3^p \in M_3$  this gives

$$(\sigma_3 - 1)w = ((p-1) \sigma_3^p + 1 - N_{G/M_3})y_3 = py_3 - N_{G/M_3}y_3.$$

Setting  $\omega = \kappa_{M_3}(w) \in H^1(M_3)$ , the  $G$ -equivariance of  $\kappa_{M_3}$  and assumption (iii) imply that

$$\begin{aligned} \sigma_3 \omega - \omega &= \kappa_{M_3}((\sigma_3 - 1)w) = -\kappa_{M_3}(N_{G/M_3}y_3) = -\text{Res}_{M_3} \kappa_G(N_{G/M_3}y_3) \\ &= -\text{Res}_{M_3} \text{Cor}_G \kappa_{M_3}(y_3) = -\text{Res}_{M_3} \text{Cor}_G \lambda_3. \end{aligned}$$

By (iii),

$$\begin{aligned} \kappa_G(N_{G/M_1}y_1 - N_{G/M_3}y_3) &= \text{Cor}_G \kappa_{M_1}(y_1) - \text{Cor}_G \kappa_{M_3}(y_3) \\ &= \text{Cor}_G \lambda_1 - \text{Cor}_G \lambda_3 = 0. \end{aligned}$$

From (ii) we obtain  $b \in A^G$  such that  $N_{G/M_1}y_1 - N_{G/M_3}y_3 = pb$ .

Next we choose  $\sigma_1 \in M_3$  such that  $G = \langle M_1, \sigma_1 \rangle$ , and denote  $M' = \langle M, \sigma_1 \sigma_3 \rangle$ . We note that  $\sigma_1, \sigma_3$  commute modulo  $M$ , so  $N_{M'/M} = \sum_{i=0}^{p-1} \sigma_1^i \sigma_3^i$  on  $A^M$ . Therefore  $N_{M'/M} = N_{G/M_3}$  on  $A^{M_3}$ , and  $N_{M'/M} = N_{G/M_1}$  on  $A^{M_1}$ . We obtain that

$$N_{M'/M}(y_3 - y_1 + b) = N_{G/M_3}y_3 - N_{G/M_1}y_1 + pb = 0.$$

By (iv),  $\hat{H}^{-1}(M'/M, A^M) = 0$ , so  $y_3 - y_1 + b = (\sigma_1 \sigma_3 - 1)t$  for some  $t \in A^M$ . Therefore

$$\begin{aligned} (\sigma_3 - 1)w &= py_3 - N_{G/M_3}y_3 = N_{M_3/M}y_3 - N_{G/M_1}y_1 + pb \\ &= N_{M_3/M}y_3 - N_{M_3/M}y_1 + pb = N_{M_3/M}(y_3 - y_1 + b) \\ &= N_{M_3/M}(\sigma_1 \sigma_3 - 1)t = \sigma_3 \sigma_1 N_{M_3/M}t - N_{M_3/M}t = (\sigma_3 - 1)N_{M_3/M}t, \end{aligned}$$

since  $\sigma_1 N_{M'/M} = N_{M'/M}$  on  $A^M$ . Thus  $w - N_{M_3/M} t \in A^{\langle M_3, \sigma_3 \rangle} = A^G$ . Taking  $\eta = \kappa_M(t) \in H^1(M)$ , we obtain using (iii) that

$$\omega - \text{Cor}_{M_3} \eta = \kappa_{M_3}(w - N_{M_3/M} t) = \text{Res}_{M_3} \kappa_G(w - N_{M_3/M} t) \in \text{Res}_{M_3} H^1(G).$$

Consequently,  $\omega \in \text{Res}_{M_3} H^1(G) + \text{Cor}_{M_3} H^1(M)$ .  $\square$

**Theorem 5.4.** *Let  $(G, A, \{\kappa_U\}_U)$  be a  $p$ -Kummer formation and let  $\chi_1, \chi_2, \chi_3 \in H^1(G)$ . Then the Massey product  $\langle \chi_1, \chi_2, \chi_3 \rangle$  is not essential.*

*Proof.* We assume that  $\langle \chi_1, \chi_2, \chi_3 \rangle$  is non-empty. By Proposition 1.1(a),  $\chi_1 \cup \chi_2 = 0 = \chi_2 \cup \chi_3$ . By Proposition 2.5, we may assume that the pairs  $\chi_1, \chi_3$  and  $\chi_2, \chi_3$  are  $\mathbb{F}_p$ -linearly independent.

Let  $M_1 = \text{Ker}(\chi_1)$ ,  $M_3 = \text{Ker}(\chi_3)$ , and  $M = M_1 \cap M_3$ , and choose  $\sigma_3 \in M_1$  such that  $G = \langle M_3, \sigma_3 \rangle$ . The exact sequence (2.1) yields  $\lambda_1 \in H^1(M_1)$  and  $\lambda_3 \in H^1(M_3)$  such that  $\text{Cor}_G \lambda_1 = \chi_2 = \text{Cor}_G \lambda_3$ . Proposition 5.3 gives rise to  $\omega \in H^1(M_3)$  such that  $\sigma_3 \omega - \omega = -\text{Res}_{M_3} \chi_2$  and  $\omega \in \text{Res}_{M_3} H^1(G) + \text{Cor}_{M_3} H^1(M)$ . By Theorem 4.2,  $0 \in \langle \chi_1, \chi_2, \chi_3 \rangle$ .  $\square$

Theorem 5.4 and Example 5.2 imply the Main Theorem.

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